

# Lecture Note

## Income and Substitution Effects

Econ 210  
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### 1 Introduction.

We have already introduced the notions of MRT coming out of the budget constraint and MRS<sup>1</sup> coming from a consumer's preferences. In our discussion of the consumer's problem we talked about how, for example, if  $MRS < MRT$  at a point on the budget line, it means that the consumer is willing to give up good 2 for good 1 at slower rate than the market. Or put another way, the market put a lower value on good 2 than the consumer at that point, which means the consumer can improve welfare by trading in good 1 for good 2 at the market rate of exchange. It is the ratio of the prices that determines the MRT and therefore determines the optimal location along the budget line for the consumer.

The problem, however, is that if we start from one solution point under a particular budget and a price changes, the MRT is not the only thing to change. One cannot change the slope of a line without changing its entire position, save one point. For the typical price change, that one point is an intercept which means, from the point of view of the previous solution point, the price change includes a significant *wealth* or *income effect*. We are going to imagine the this change in the budget line from a price change taking place in two steps: (i) a rotation to the new MRT pivoting through the original choice (and therefore stripping away the income effect) followed by (ii) a parallel shift from there out (or in) to the new budget line - which adds back the income effect we stripped away in the rotation.

First the rotation pivoting through the original choice ... Imagine standing on our "hill of happiness" at the original solution point. You are right on top of the fence-line (budget line) at its highest point on the hill. Now somebody changes the angle at which that fence goes across the side of hill by rotating it around the point you are standing on. With this

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<sup>1</sup>As mentioned earlier, there are always 2 MRTs [*2 MRSs*] in a two-good world, there is the amount of good 1 one must [*is willing to*] give up to get another unit of good 2 and there is the amount of good 2, one must [*is willing to*] give up to get another unit of good 1. These two rates of exchanges should be inverses of each other as long as they are both finite and non-zero. Unless stated otherwise, I mean the latter one, so that  $MRT = \frac{p_1}{p_2}$  and  $MRS = \frac{\partial u / \partial x_1}{\partial u / \partial x_2}$  when preferences can be represented by a utility function.

new fence you *could* stay in the same place - since you are standing at the pivot point. However, the new angle of the fence-line has exposed some newly accessible points of higher elevation on the hill. If the price change really produced a new fence line pivoting around your old solution point, this is how it would feel and you would move in the direction of the good whose relative price fell. Second the parallel shift. **However, in fact, it was not only the angle of the fence that changed.** Not only did the MRT change, the fence's position also was shifted down the hill (for a price increase).

In this note, we will decompose the demand response to price changes into *substitution effects* and *income effects*. The substitution effect is the direction and distance a consumer would move if the budget line changed to its new slope (MRT) dictated by the new prices, but did so by pivoting on the original demand solution. This hypothetical intermediate budget line is the *Slutsky compensated* budget line and the highest point on the hill along it is the *compensated choice*. The *income effect* is the remainder of the response to the price change. It is the direction and magnitude of the response from the compensated budget line to the new budget line.

## 2 The Total Effect.

*“Keep Your Grasp on the Obvious Firm”*  
- Marv Johnson

Let  $x_1(p_1, p_2, m)$  be the demand for good 1 when the price of good 1 is  $p_1$ , the price of good 2 is  $p_2$  and the consumer's income level is  $m$ . Consider a price change in good 1 from  $p_1^0$  to  $p_1^1$ . This may be a price-increase or a price decrease. The *total effect* of the price change on the demand for good 1 is defined as follows.

$$\text{Total Effect} = x_1(p_1^1, p_2, m) - x_1(p_1^0, p_2, m)$$

Since  $x_1(p_1^1, p_2, m)$  is the demand under the *new* price of good 1 we will call the **New Demand**. By the same token, we will refer to  $x_1(p_1^0, p_2, m)$  as the **Old Demand**. And so, in words, the *total effect* is simply

$$\text{Total Effect} = [\text{New Demand}] - [\text{Old Demand}]$$

Note that if the good is *ordinary* (a.k.a. not Giffen), and the price change is a price decrease, the total effect will be a positive quantity. If the price change is price increase, the total effect will be a negative quantity.

### 3 Slutsky Compensated Demand

The first step in our decomposition is to construct a *compensated* budget,  $(p_1^1, p_2, m^c)$ . It is extremely important to bear in mind that this budget line is **hypothetical**. That is, it is all in our heads; The consumer never actually experienced this budget at any point in the price transition even though we are going to ask what would happen if she did.

In this *compensated* budget, the prices will be  $p_1^1$  and  $p_2$ . In other words, the MRT under this hypothetical budget will be the **new** MRT, since  $p_1^1$  is the new price of good 1. The difference between the actual new budget and this hypothetical compensated budget is the income level. In the compensated budget, we will provide exactly the income required to purchase the **old** consumption bundle - that is, the consumption bundle demanded under the **old** budget. Therefore the compensated income level,  $m^c$  is given by.

$$m^c = p_1^1 x_1(p_1^0, p_2, m) + p_2 x_2(p_1^0, p_2, m)$$

In words, the compensated level of income is the **new** price of good 1 times the quantity of good 1 demanded under the **old** price of good 1 plus the money required to purchase the quantity of good 2 demanded under the **old** price of good 1. <sup>2</sup>

Another way to calculate the compensating level of income is as follows

$$m^c = m + (p_1^1 - p_1^0) x_1(p_1^0, p_2, m)$$

Thought of in this way, the compensating level of income, for a price increase, is simply the old level of income plus the additional money required to continue purchasing the **old** demand of good 1. For a price decrease, it is the old level of income minus the saving experienced purchasing the *old* quantity of good 1 due to the price decrease.

**Problem.** Prove that these two formulas for the compensating level of income amount to the same thing. I.e. show that  $m^c = p_1^1 x_1(p_1^0, p_2, m) + p_2 x_2(p_1^0, p_2, m) = m + (p_1^1 - p_1^0) x_1(p_1^0, p_2, m)$

If you were to draw the compensated budget line, it would intersect the **old demand point** - that is, the optimal consumption bundle under the old price - and have the **slope of the new budget line**.

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<sup>2</sup>Note that we are talking about the entire consumption bundle demanded under the old set of prices. It is important to realize that just because we are only talking about a change in the price of good 1, that doesn't mean that the demand for good 2 remains constant. Therefore when formulating the compensating level of income, we have to specify  $x_2(p_1^0, p_2, m)$ , the old demand of good 2.

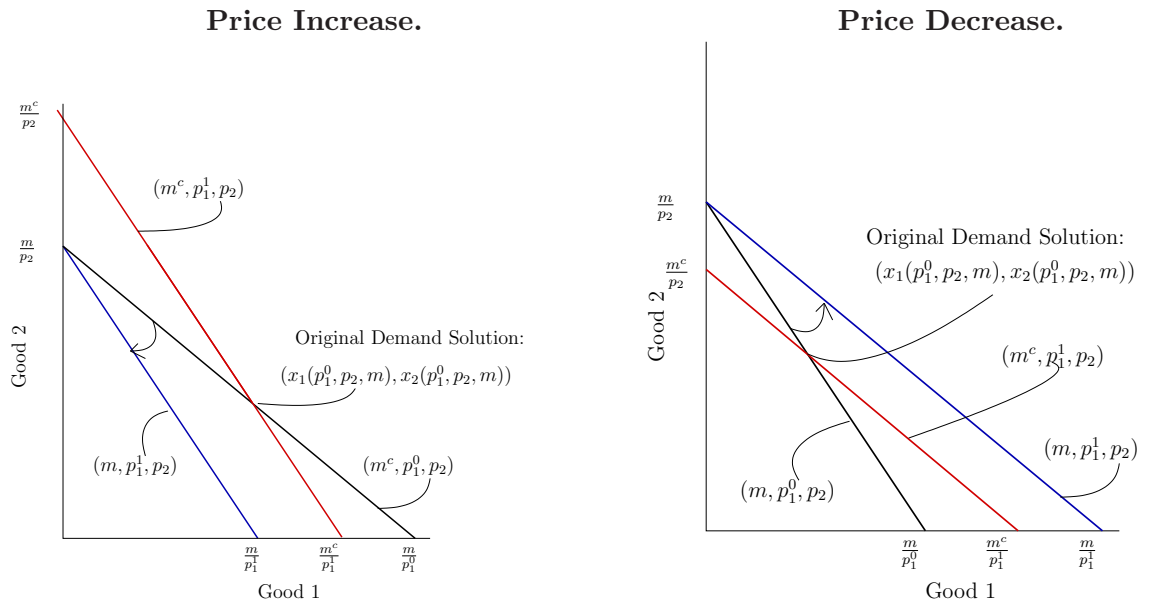
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Drawing A Compensated Budget Line

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The pictures show changes in the price of good 1. In each case, the black budget line is the original budget line. The new budget line is shown in blue. The compensated budget line intersects the old demand point but reflects in its slope the new price of good 1 - steeper in the case of a price increase, shallower in the case of a decrease.

The *compensated demand* for good 1,  $x_1(p_1^1, p_2, m^c)$ , is the good 1 component of the solution to the consumer's problem if the consumer face the compensated budget.

$$\begin{aligned}
 (x_1(p_1^1, p_2, m^c), x_2(p_1^1, p_2, m^c)) &= \underset{(x_1, x_2)}{\operatorname{argmax}} u(x_1, x_2) \\
 & \text{s.t. } p_1^1 x_1 + p_2 x_2 < m^c
 \end{aligned}$$

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The pictures on the next page show how to draw compensated demand.

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<sup>3</sup>argmax means the argument which maximizes the expression. Hence in conjunction with the budget constraint,  $\underset{(x_1, x_2)}{\operatorname{argmax}} u(x_1, x_2)$  refers to the consumption bundle affordable under the compensated budget that maximizes the utility function.

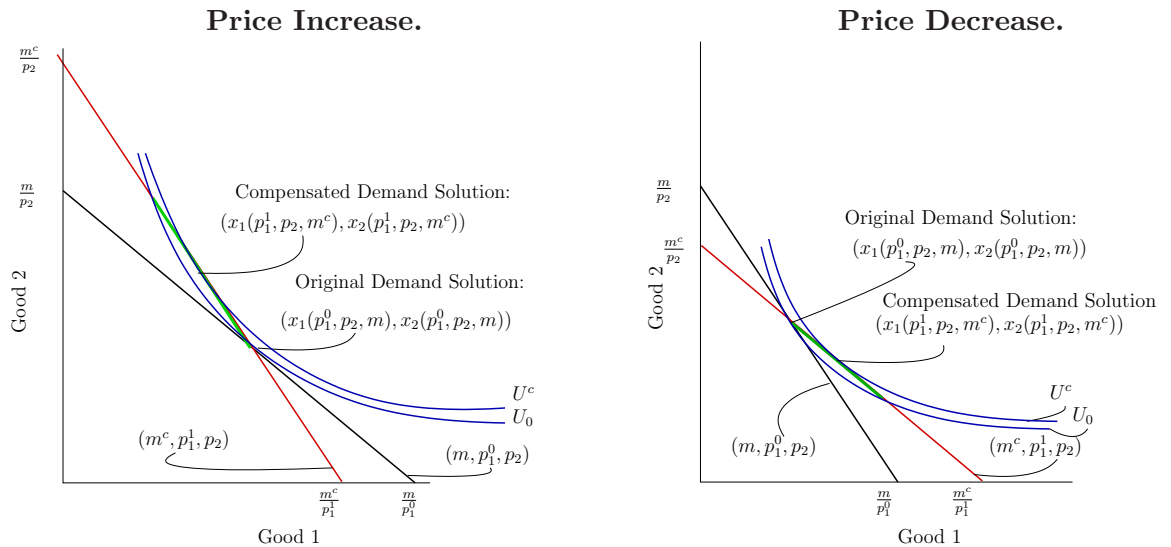
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## Finding Compensated Demand

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The pictures show how to find compensated demand. In each case, the black budget line is the original budget line. The compensated budget line (in red) intersects the old demand point but reflects in its slope the new price of good 1 - steeper in the case of a price increase, shallower in the case of a decrease.

There are several things to note about compensated demand in the pictures above.

- The “Law of Compensated Demand”. The necessary directional changes in compensated demand is a result of the convexity of preferences.
  - The optimal consumption bundle for a compensated price increase in good 1 *must* be up and to the left (less good 1, more good 2) of the original solution point.
  - The optimal consumption bundle for a compensated price decrease in good 1 *must* be down and to the right (more good 1, less good 2) of the original solution point.
- The bold green sections of the compensated budget line show where the compensated demand solution must lie given the original demand solution and its associated indifference curve.
- $u^c > u_0$ . Slutsky Compensation is “over compensation” in the sense of welfare. Note that in both pictures the utility level one would reach if faced with the compensated budget line would be at least as high and typically higher than the level of utility

achieved under the original budget. There is another notion of compensation, known as Hicksian Compensation, which corrects for this.

## 4 Income and Substitution Effects

The substitution effect of a price change describes the transition between the original demand solution and the compensated demand solution. The *income effect* describes the direction and magnitude of the transition from compensated demand to new demand.

$$\begin{aligned}\text{Subst.Effect} &= \text{Compensated Demand} - \text{Old Demand} \\ \text{IncomeEffect} &= \text{New Demand} - \text{Compensated Demand}\end{aligned}$$

If we look only at the good 1 component of the decomposition we have

$$\begin{aligned}\text{Subst. Effect} &= x_1(p_1^1, p_2, m^c) - x_1(p_1^0, p_2, m) \\ &\quad (\text{Comp'd Demand}) \quad -(\text{Old Demand}) \\ \text{Income Effect} &= x_1(p_1^1, p_2, m) - x_1(p_1^1, p_2, m^c) \\ &\quad (\text{New Dmd}) \quad -(\text{Comp'd Dmd})\end{aligned}$$

Notice that the sum of income and substitution effects is the total effect. This observation is known as Slutsky's Equation

$$\begin{aligned}\text{Total Effect} &= \text{Income Effect} + \text{Subst Effect} \\ [\text{New Demand} - \text{Old Demand}] &= [(\text{New Dmd}) - (\text{Comp'd Dmd})] + [(\text{Comp'd Dmd}) - (\text{Old Dmd})] \\ [x_1(p_1^1, p_2, m) - x_1(p_1^0, p_2, m)] &= [x_1(p_1^1, p_2, m) - x_1(p_1^1, p_2, m^c)] + [x_1(p_1^1, p_2, m^c) - x_1(p_1^0, p_2, m)]\end{aligned}$$

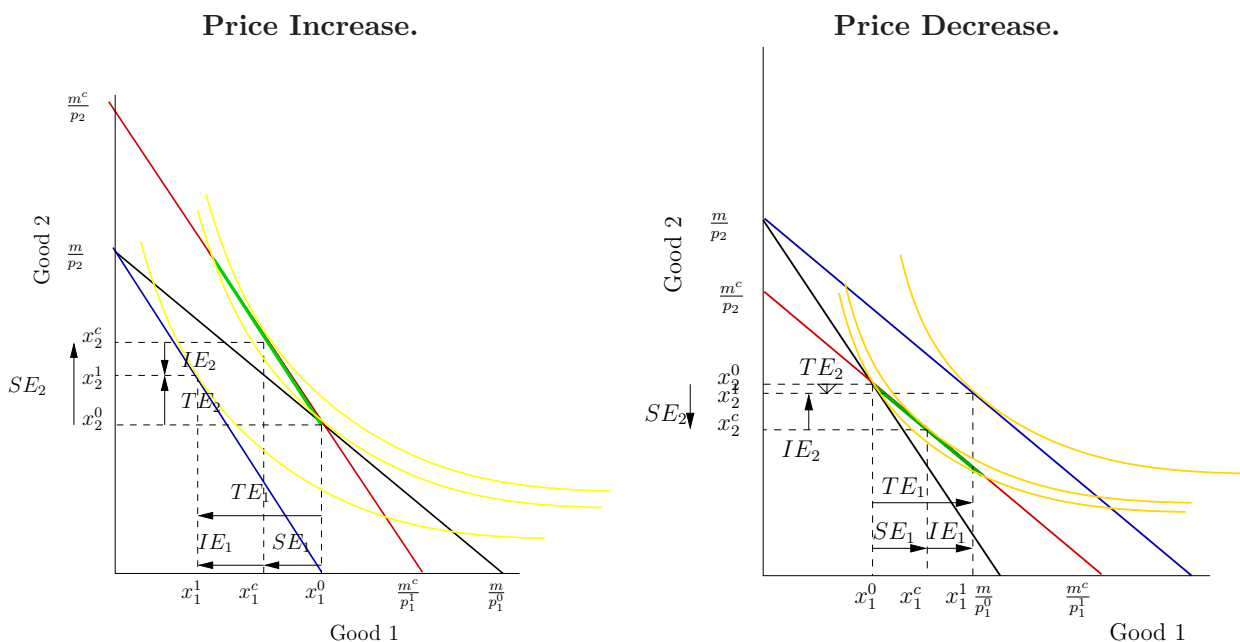
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Identifying Income and Substitution Effects.

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The pictures show how to identify income, substitution and total effect for the case of both a price increase and price decrease. In each case, original budget line is shown is black, the compensated in red and new in blue. What type of goods are these? (Normal, Inferior-not-Giffen, or Giffen). How can you tell?

Further Exercises.

1. Try drawing all the pictures. I count twenty cases.<sup>4</sup>

Good on Vertical Axis (VG) is	N	N	IO	N	G
Good on Vertical Axis (HG) is	N	IO	N	G	N
Price of VG Increases					
Price of VG Decreases					
Price of HG Increases					
Price of HG Decreases					

Normal, **IO**  $\equiv$  Inferior Ordinary, **G**iffen.

Review Question: Why no (IO, IO ) and (G,G) columns in this table?

2. Decompose price changes for Cobb-Douglas, PS and PC preferences.

<sup>4</sup>Well, somewhat fewer true distinct cases, since i'm double counting cases by simply interchanging whats on the horizontal and vertical axes, but extra practice won't hurt.